

On Trading American Put Options with Interactive Volatility

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Abstract

We introduce a simple stochastic volatility model, which takes into account hitting times of the asset price, and study the optimal stopping problem corresponding to a put option whose time horizon (after the asset price hits a certain level) is exponentially distributed. We obtain explicit optimal stopping rules in various cases one of which is interestingly complex because of an unexpectedly disconnected continuation region. Finally, we discuss in detail how these stopping rules could be used for trading an American put when the trader expects a market drop in the near future.

KEY WORDS optimal stopping, stochastic volatility model, regime switching, American option

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1 Introduction and Results

This paper deals with an optimal stopping problem which is motivated by option trading.

First, we introduce a simple short term model for the price of an asset which is able to capture some aspects of the so-called leverage effect, and, second, under such a model, we predict value and exercise time of a perpetual American put written on this asset.

The leverage effect refers to the phenomenon that, typically, decreasing asset prices are accompanied by rising volatility. We will not argue about whether the leverage effect is a true phenomenon or not. We rather treat it as an observed phenomenon which has been discussed in many papers since the mid 1970s when Black, [1], gave a well received macroeconomic explanation. Since this effect has been observed, risk-seeking market participants might want to take advantage of it.

However, other effects can superimpose a possible leverage effect. For example, a decreasing stock price after a negative earning report usually goes along with falling volatility as uncertainty decreases after an announced event. Hence, the decision to bet on a combination of falling prices and rising volatility requires a careful analysis of relevant market conditions which is left to the acting market participants.

The market participant we have in mind is an option trader who has made this decision and plans to go long on an American put. The rationale behind going long on an American put when betting on a leverage effect is twofold; falling prices and increasing volatility would both raise put prices. But, if the trader wants to understand the risk of such a betting strategy before entering the trade, they should create a model for the price $(S_t, t \geq 0)$ of the asset underlying the American put which, first,

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is simple enough, second, is able to capture key features of the trader's preferences for the future and, third, has enough parameters to control the probabilities of different scenarios of future prices.

The model we propose can heuristically be described as follows:

- the price S_t , $t \geq 0$, behaves like a geometric Brownian motion with volatility parameter σ_0 and trend μ_0 until it hits a critical level $s_0 \ll s$ where s is the present price;
- when hitting the critical level, the volatility parameter steps up to σ_1 and the trend of the stock changes to μ_1 ;
- this 'excited' state lasts for a period of length T which is exponentially distributed with rate λ ;
- finally the price is frozen at its value taken when the exponential time T has expired.

Note that the above bullet points characterise a type of stochastic volatility model which has not been discussed in the literature, yet. We call such a stochastic volatility *interactive volatility* to emphasise its dependence on hitting times of the price process.

Remark 1.1. (i) The above model supposes $S_t = S_{t \wedge (\tau_{s_0} + T)}$ for $t \geq 0$ where

$$\tau_a \stackrel{\text{def}}{=} \inf\{t \geq 0 : S_t \leq a\} \quad (1.1)$$

for given price levels a . The reason for freezing the price at $\tau_{s_0} + T$ is that this time span is considered the time horizon of the trade. Studying a perpetual American put under this model easily reveals that the put should be optimally exercised at the random time $\tau_{s_0} + T$, latest—see Remark 1.3(ii).

- (ii) The notation $s_0 \ll s$ is used to emphasise that the difference $s - s_0$ between the present price of the asset and the critical level should be chosen big enough. The size of $s - s_0$ determines the strength of the market's drop which causes the regime change from volatility σ_0 to σ_1 according to the leverage effect.
- (iii) A reasonable choice for σ_0 would be the implied volatility at present time of the traded American put the trader wants to long. Now recall that

$$\log S_t = \log s + (\mu_0 - \frac{\sigma_0^2}{2})t + \sigma_0 \widehat{B_t}^{\text{Brownian motion}}$$

is assumed to hold for $t \in [0, \tau_{s_0})$. Hence, for fixed σ_0 , the choice of μ_0 determines the distribution of the hitting time τ_{s_0} . To meet the preferences of the trader of a market-fall in the near future, μ_0 should be chosen sufficiently small to decrease the probability of large values of τ_{s_0} . But, the trader should also analyse the optimal stopping problem for larger values of μ_0 , that is, they should analyse their position under the assumption they are wrong and the probability of a market-fall in the near future is rather small.

- (iv) For $t \in [\tau_{s_0}, \tau_{s_0} + T)$, which is the final period of the trade, the trader assumes

$$\log S_t = \log s_0 + (\mu_1 - \frac{\sigma_1^2}{2})(t - \tau_{s_0}) + \sigma_1(B_t - B_{\tau_{s_0}}).$$

The choice of the parameters σ_1, μ_1 reflects the trader's view on the strength of the regime change triggered by the leverage effect, and hence this choice is more or less subject to both the trader's experience and their understanding of the market's history.

- (v) Using an exponential time T for modelling the time span of the impact of the leverage effect keeps the model simple enough. It is also assumed that T is independent of what has happened before τ_{s_0} . The parameter λ should be big enough to ensure that, on average, the time span of the new volatility regime is of the order of days and not weeks. If both τ_{s_0} and T are on average rather short then the whole trade's time horizon is likely to be less than the time to maturity of traded American puts.
- (vi) Following the suggestions made in items (iii),(iv) above, the trader's reasoning behind choosing μ_0, μ_1 has nothing to do with the market's rate of interest during the time span of their trade. Thus, the model's underlying probability measure should be considered a guess of the real-world measure rather than a pricing measure. Working out the optimal exercise time of the perpetual American put in the context of this model gives the trader an indication of when to exit a trade they entered in accordance with their own preferences for the future. The value of the put under the model is mainly used for finding the optimal exercise time, and should NOT be confused with the price of a traded put.
- (vii) Our analysis can be used to motivate the choice of a traded American put with reasonable strike level and time to maturity for the purpose of betting on a combination of falling prices and rising volatility—see Section 3 for the details.

The proposed model has features of a Markov chain regime switching volatility model as the excited state, when the volatility is σ_1 , lasts for an exponential time. But, at the end of this exponential time, instead of moving into a state which corresponds to another volatility level, the Markov chain moves into an absorbing state. So, for the second and final period of the trade, the model can be regarded as a degenerated Markov chain regime switching volatility model. We will comment on Markov chain regime switching volatility models in Remark 1.7(ii) below.

For the initial period of the trade, the model is different to a Markov chain regime switching volatility model as the system does not enter the excited state following the move of a Markov chain. Instead, it enters the excited state according to if the price of the asset has fallen to the critical level s_0 or not, that is, according to how the price of the asset has behaved in the past.

To achieve Markovianity, we add a process $(Y_t, t \geq 0)$ for screening whether the price S_t has already fallen to the critical level s_0 or not. To fully describe the dynamics of S_t , we also add a process $(\eta_t, t \geq 0)$ which is an absorbing Markov chain screening the length of the excitation.

Technically, we work with a strong Markov process $(S, Y, \eta) = (S_t, Y_t, \eta_t, t \geq 0)$ on a family of probability spaces $(\Omega, \mathcal{F}, \mathbf{P}_{s,y,i}, (s, y, i) \in A)$, where

$$A \stackrel{\text{def}}{=} \left[(s_0, \infty) \times \{0\} \times \{1\} \right] \cup \left[(0, \infty) \times \{1\} \times \{0, 1\} \right]$$

is considered a subset of the topological space $(0, \infty) \times \{0, 1\} \times \{0, 1\}$ equipped with the product topology.

The generator of this process is formally defined by

$$\begin{aligned} Lf(s, 0, 1) &= \mu_0 s \partial_1 f(s, 0, 1) + \frac{1}{2} \sigma_0^2 s^2 \partial_1^2 f(s, 0, 1) & \text{for } s \in (s_0, \infty), \\ Lf(s, 1, 1) &= \mu_1 s \partial_1 f(s, 1, 1) + \frac{1}{2} \sigma_1^2 s^2 \partial_1^2 f(s, 1, 1) + \lambda [f(s, 1, 0) - f(s, 1, 1)] & \text{for } s \in (0, \infty), \\ Lf(s, 1, 0) &= 0 & \text{for } s \in (s, \infty). \end{aligned}$$

It is considered an unbounded operator on the space $C_b(\bar{A})$ of all bounded continuous functions on \bar{A} whose domain consists of all $f \in C_b(\bar{A})$ satisfying both $f(s_0, 1, 1) - f(s_0, 0, 1) = 0$ and $Lf \in C_b(\bar{A})$ where the latter condition is understood in the sense of Schwartz distributions.

Remark 1.2. (i) The condition $f(s_0, 1, 1) - f(s_0, 0, 1) = 0$ is a discrete Neumann boundary condition, and this boundary condition establishes an interaction between the states $(s_0, 0, 1)$ and $(s_0, 1, 1)$ leading to a jump of the process Y_t when the price S_t reaches s_0 .

(ii) As a consequence, for all $s > s_0$, under $\mathbf{P}_{s,0,1}$, it holds that $Y_t = \mathbf{1}_{[\tau_{s_0}, \infty)}(t)$, $t \geq 0$, and $\eta_t = 1$, $t \leq \tau_{s_0}$, whereas, for all $s > 0$, under $\mathbf{P}_{s,1,1}$, it holds that $Y_t = 1$, $t \geq 0$, and $(\eta_t, t \geq 0)$ is an independent two-states continuous-time Markov chain starting from one and absorbed at zero with rate λ .

(iii) Combining (ii) and $Lf(s, 1, 0) = 0$ for $s > 0$ yields

$$\mathbf{P}_{s,y,i} \left(\{ S_t = S_{t \wedge \tau_{\eta,0}} \text{ for all } t \geq 0 \} \right) = 1 \quad \text{for all } (s, y, i) \in A,$$

where

$$\tau_{\eta,0} \stackrel{\text{def}}{=} \inf \{ t \geq 0 : \eta_t = 0 \}, \quad (1.2)$$

that is, $\tau_{\eta,0}$ plays the role of what was called $\tau_{s_0} + T$ in Remark 1.1(i).

(iv) Taking into account the other defining properties of the generator L , the S -component of the process (S, Y, η) started at $s > s_0$ has, under $\mathbf{P}_{s,0,1}$, the same law as the price process discussed in items (iii) and (iv) of Remark 1.1. Note that we could have worked with the process (S, Y) killed at rate λ after the jump of Y , instead. But, as explained in Remark 1.7(ii) below, using an extra component like η has the advantage that we can apply results on optimal stopping in the context of Markov chain regime switching models.

(v) Since (S, Y, η) is strong Markov, the filtration $(\mathcal{F}_t, t \geq 0)$ generated¹ by (S, Y, η) is right-continuous, and, by obvious reasons, this filtration coincides with the smallest right-continuous filtration which contains the universal augmentation of $(\sigma(S_u : u \leq t), t \geq 0)$.

All in all, we have established a probability model for the prospective prices of an asset which induces the wanted features laid out in the four bullet points on page 2.

Next, under this model, we will study value and optimal exercise time of a perpetual American put contract written on this asset. Using our probability model, such a put's value function takes the form

$$V(s, y, i) \stackrel{\text{def}}{=} \sup_{\tau \geq 0} \mathbf{E}_{s,y,i} [e^{-\alpha\tau} (K - S_\tau)^+] \quad \text{for } (s, y, i) \in A, \quad (1.3)$$

where the supremum is taken over all stopping times with respect to the filtration $(\mathcal{F}_t, t \geq 0)$.

Remark 1.3. (i) The discount rate α refers to the rate of return of an investment the trader considers more or less riskless during the time interval of the trade. As explained in items (iii) and (v) of Remark 1.1, under the future preferences of the trader, this time interval is supposed to be rather short on average, and hence choosing α to be constant is a good approximation.

(ii) Note that

$$V(s, y, i) = \sup_{\tau \leq \tau_{\eta,0}} \mathbf{E}_{s,y,i} [e^{-\alpha\tau} (K - S_\tau)^+]$$

because $S_t = S_{t \wedge \tau_{\eta,0}}$, $t \geq 0$, implies $e^{-\alpha\tau} (K - S_\tau)^+ \leq e^{-\alpha\tau_{\eta,0}} (K - S_{\tau_{\eta,0}})^+$ on $\tau \geq \tau_{\eta,0}$. Hence, under our model, the perpetual put should be exercised at $\tau_{\eta,0} = \tau_{s_0} + T$, latest.

¹Here, 'filtration generated by (S, Y, η) ' refers to the universal augmentation of the filtration $(\sigma(S_u, Y_u, \eta_u : u \leq t), t \geq 0)$ see Section 2.7.B of [5] for a good account on universal filtrations.

First recall the results for perpetual American put options as obtained in [7] in the context of geometric Brownian motion.

Theorem 1.4. *Given on a family of probability spaces $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}_s, s > 0)$, let $(\tilde{S}_t, t \geq 0)$ be the Feller process whose generator is the closure of*

$$\tilde{L}f = \mu_0 s f' + \frac{1}{2} \sigma_0^2 s^2 f'', \quad f \in C_b^2((0, \infty)).$$

Then, the value function

$$\tilde{V}(s) \stackrel{\text{def}}{=} \sup \left\{ \tilde{\mathbf{E}}_s [e^{-\alpha \tilde{\tau}} (K - \tilde{S}_{\tilde{\tau}})^+] : \tilde{\tau} \text{ stopping time with respect to } (\tilde{S}_t, t \geq 0) \right\}$$

is given by

$$\tilde{V}(s) = \begin{cases} K - s & : s \in (0, b_0], \\ (K - b_0) \left(\frac{s}{b_0} \right)^{\gamma^-} & : s \in (b_0, \infty), \end{cases} \quad (1.4)$$

where $b_0 = -\gamma^- K / (1 - \gamma^-)$, and γ^- stands for the negative root of the quadratic equation

$$\frac{1}{2} \sigma_0^2 \gamma^2 + (\mu_0 - \frac{1}{2} \sigma_0^2) \gamma - \alpha = 0. \quad (1.5)$$

Remark 1.5. (i) The above value function \tilde{V} satisfies

$$0 = \mu_0 s \tilde{V}'(s) + \frac{1}{2} \sigma_0^2 s^2 \tilde{V}''(s) - \alpha \tilde{V}(s) \quad \text{for } s > b_0$$

subject to

$$\tilde{V}(b_0) = K - b_0, \quad \tilde{V}'(b_0) = -1, \quad \lim_{s \rightarrow \infty} \tilde{V}(s) = 0.$$

- (ii) If $\mu_0 - \frac{1}{2} \sigma_0^2 > 0$, then there is no (finite) optimal stopping time at which the value function \tilde{V} can be attained. But $\tilde{\tau}_{b_0}$ is a Markov time at which \tilde{V} given in Theorem 1.4 is attained when setting $\tilde{\mathbf{E}}_s [e^{-\alpha \tilde{\tau}_{b_0}} (K - \tilde{S}_{\tilde{\tau}_{b_0}})^+]$ to be zero on $\{\tilde{\tau}_{b_0} = \infty\}$. In all further cases below, attaining a value function at a possibly infinite Markov time will be understood as above, since all considered value functions vanish at infinity.

The next theorem presents the main result of this paper.

Theorem 1.6. *Recall (1.1), (1.2), and Remark 1.1(ii) for the purpose of s_0 , and Remark 1.5(ii) for the meaning of b_0 . Let γ^+ (γ^-) denote the positive (negative) root of equation (1.5). The following cases completely describe the value function given by (1.3).*

- (i) *In the trivial case,*

$$V(s, 1, 0) = (K - s)^+ \quad \text{for all } s > 0,$$

and the optimal stopping time is 0.

(ii) Let β^+ (β^-) denote the positive (negative) root of the quadratic equation

$$\frac{1}{2}\sigma_1^2\beta^2 + (\mu_1 - \frac{1}{2}\sigma_1^2)\beta - (\alpha + \lambda) = 0. \quad (1.6)$$

Then, there exists a smooth function $h : (0, \infty) \rightarrow \mathbb{R}$ such that

$$V(s, 1, 1) = \begin{cases} K - s & : s \in (0, b_1], \\ c_1 s^{\beta^+} + c_2 s^{\beta^-} + h(s) & : s \in (b_1, K], \\ d_2 s^{\beta^-} & : s \in (K, \infty), \end{cases}$$

where the coefficients c_1, c_2, d_2 and the stopping level b_1 are obtained by solving the equations (2.4) on page 9. The finite optimal stopping time is $\tau_{b_1} \wedge \tau_{\eta, 0}$.

(iii) If one of the conditions

$$(a) \ b_0 > s_0 \text{ and } V(s_0, 1, 1) \geq (K - b_0)(s_0/b_0)^{\gamma^-},$$

$$(b) \ b_0 \leq s_0 \text{ and } V(s_0, 1, 1) > (K - s_0)^+,$$

$$(c) \ b_0 \leq s_0 < K \text{ and } V(s_0, 1, 1) = (K - s_0),$$

is satisfied, then

$$V(s, 0, 1) = V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} \quad \text{for } s > s_0.$$

This value function is attained at the possibly infinite Markov time $\tau_{b_1} \wedge \tau_{\eta, 0}$, if (a), (b), and τ_{s_0} , if (c).

(iv) If $b_0 > s_0$ and $K - s_0 \leq V(s_0, 1, 1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, then

$$V(s, 0, 1) = \begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : s \in (s_0, b_*), \\ K - s & : s \in [b_*, b_0] \cap (s_0, \infty), \\ (K - b_0) \left(\frac{s}{b_0} \right)^{\gamma^-} & : s \in (b_0, \infty), \end{cases}$$

where² $b_* = s_0$ if $K - s_0 = V(s_0, 1, 1)$, and e_1^*, e_2^*, b_* as obtained in the proof of Lemma 2.3 on page 14, otherwise. The value function is attained at the possibly infinite Markov time $\tau_{[b_*, b_0], 0} \wedge \tau_{b_1} \wedge \tau_{\eta, 0}$ where

$$\tau_{[a, b], 0} \stackrel{\text{def}}{=} \inf\{t \geq 0 : S_t \in [a, b], Y_t = 0\}$$

for levels $0 < a < b$.

Remark 1.7. (i) The contribution of this paper consists in the two non-trivial cases (iii) and (iv) but also in the review of the known case (ii)—see Remark 1.7(iii+iv) below. Case (iv) is mathematically most interesting because of an unexpectedly disconnected continuation region. As explained in Remark 1.1(vi), only the optimal stopping times are of true relevance for the trader. Note that there are no further sub-cases than those mentioned under (iii) and (iv). The restriction to $s_0 < K$ in case (iii)(c) follows from the fact that $V(s_0, 1, 1) > 0$ if $s_0 \geq K$. We will discuss numerical examples for all sub-cases in Section 3.

²Here $(s_0, b_*) = \emptyset$ by definition when $b_* = s_0$.

- (ii) Both cases (iii) and (iv) are based on case (ii), that is, on the solution of the optimal stopping problem under the measures $\mathbf{P}_{s,1,1}$, $s > 0$. Under these measures, the price process $(S_t, t \geq 0)$ satisfies the stochastic differential equation

$$dS_t = \mu_1 \eta_t S_t dt + \sigma_1 \eta_t S_t dB_t$$

which is a consequence of what was discussed in Remark 1.1(iv) and (ii)+(iii) of Remark 1.2. Note that the above stochastic differential equation describes a Markov modulated geometric Brownian motion which has extensively been used in the literature as a simple Markov chain regime switching volatility model. However, there are just two important papers on pricing perpetual American puts in the context of such a regime switching model: one by Guo/Zhang, [3], treating the case of two-states Markov chains, and another by Jobert/Rogers, [4], treating the general case of Markov chains with finitely many states. Buffington/Elliott's, [2], work on finite time horizon American puts can be considered an extension of the ideas behind [3]. The explicit form of the value function given in case (ii) is a special case of Guo/Zhang's result when the Markov chain is degenerated. The function $h : (0, \infty) \rightarrow \mathbb{R}$ can be any solution to the first of the two differential equations above (2.2) on page 8, for example,

$$h(s) = -\frac{\lambda s}{\alpha + \lambda - \mu_1} + \frac{\lambda K}{\alpha + \lambda} \quad \text{and} \quad h(s) = \frac{\lambda s \log s}{\alpha + \lambda + \sigma_1^2/2} + \frac{\lambda K}{\alpha + \lambda}$$

for $\alpha + \lambda \neq \mu_1$ and $\alpha + \lambda = \mu_1$, respectively.

- (iii) To prove that the explicit expression given for $V(\cdot, 1, 1)$ in case (ii) is indeed the value function requires a verification argument. Denoting this explicit expression by $V^*(\cdot, 1, 1)$, the standard method of verification would be to verify the properties (v1),(v2),(v3) as given at the beginning of Section 2.2.1, but with respect to the measures $\mathbf{P}_{s,1,1}$, $s > 0$. Following [3], one realises that the authors do not verify but assume (v3) in their Theorem 3.1. Furthermore, their proof of (v2) is incomplete as they only justify $(\alpha - L)V^*(\cdot, 1, 1) \leq 0$ inside of the continuation region. Outside of the continuation region, that is, when $V^*(\cdot, 1, 1)$ coincides with $(K - \cdot)^+$, the validity of $(\alpha - L)V^*(\cdot, 1, 1) \leq 0$ would depend on the value of b_1 , and this issue was not addressed in [3]. Finally they also assume $\mu_1 \geq 0$ which is an assumption we might want to avoid as explained in Remark 1.1. All in all, the verification of the value function given in [3] is incomplete and their assumptions are too restrictive for our purpose, and that's why we decided to address this verification in the Appendix.
- (iv) We also checked the verification arguments given in the proof of [4, Prop.2]. Here, the authors in particular demonstrate $(\alpha - L)V^*(\cdot, 1, 1) \leq 0$ outside of the continuation region but their proof requires $(e^{-\mu_1 t} S_t, t \geq 0)$ to be a supermartingale which restricts the choice of μ_1 to $\mu_1 \leq r$. We cannot follow their proof of (v3) as it looks to us as if they applied Doob's optional sampling theorem using an unbounded stopping time. Furthermore, we do not see why, without using further arguments, the non-linear equations mentioned in [4, Problem 2] should have unique solutions (see Remark A.1 in the Appendix for further details).

2 Proofs

Both cases (i) and (ii) of Theorem 1.6 assume $y = 1$ which leads to a special version of the results obtained in [3] where American puts were priced in the context of a two-states Markov chain volatility model. In our case, the Q -matrix of the corresponding Markov chain is degenerated.

Therefore, in the next section, we only sketch the proof of case (ii). But we give enough details to put the notation used in Theorem 1.6(ii) into context. However, recall Remark 1.7(iii) where we explained that the verification of the value function in [3] was incomplete. The corresponding details can be found in the Appendix.

2.1 Proof of Theorem 1.6 (i) and (ii)

For all $s > 0$, and any stopping time τ ,

$$\mathbf{E}_{s,1,i}[e^{-\alpha\tau}(K - S_\tau)^+] \leq \mathbf{E}_{s,1,i}[e^{-\alpha(\tau \wedge \tau_{\eta,0})}(K - S_{\tau \wedge \tau_{\eta,0}})^+] \quad \text{for } (s, i) \in (0, \infty) \times \{0, 1\}$$

since the process $(S_t, t \geq 0)$ is stopped at $\tau_{\eta,0}$. Hence $V(s, 1, 0) = (K - s)^+$ with optimal stopping time 0 proving (i).

For showing (ii), recall that $Y_t = 1$ for all $t \geq 0$ a.s. when starting the dynamics from any $s > 0$, $y = i = 1$. Now, assume that the stopping region takes the form

$$(0, b_1] \times \{1\} \times \{1\} \cup (0, \infty) \times \{1\} \times \{0\}$$

when starting from $s > 0$, $y = i = 1$ where b_1 is an unknown stopping level. Then, by [9, Theorem 2.4] for example, if V is lower semi-continuous, the value function $V(s, 1, 1)$ would be attained at

$$\tau^* = \tau_{\eta,0} \wedge \tau_{b_1}, \quad (2.1)$$

and $(e^{-\alpha(\tau^* \wedge t)}V(S_{t \wedge \tau^*}, Y_{t \wedge \tau^*}, \eta_{t \wedge \tau^*}), t \geq 0)$ would be a $\mathbf{P}_{s,1,1}$ -martingale.

We want to use this martingale property to derive equations for both V and b_1 . Assume for now that V has even more regularity and a generalised Itô's formula (see Remark 2.2 below) can be applied to obtain

$$e^{-\alpha(t \wedge \tau^*)}V(S_{t \wedge \tau^*}, Y_{t \wedge \tau^*}, \eta_{t \wedge \tau^*}) = V(S_0, Y_0, \eta_0) + \int_0^{t \wedge \tau^*} e^{-\alpha u}(L - \alpha I)V(S_u, Y_u, \eta_u)du + M_{t \wedge \tau^*}$$

for all $t \geq 0$ a.s. where M stands for a local martingale and I denotes the identity operator.

Of course, for the above left-hand side to be a martingale, the integral on the right-hand side must vanish. Using both the specific form of L as given on page 3 and (i) proven above, a sufficient condition for this integral to vanish is

$$\begin{cases} 0 = \mu_1 s \partial_1 V(s, 1, 1) + \frac{1}{2} \sigma_1^2 s^2 \partial_{11} V(s, 1, 1) + \lambda(K - s) - (\alpha + \lambda)V(s, 1, 1) & \text{for } s \in (b_1, K) \\ 0 = \mu_1 s \partial_1 V(s, 1, 1) + \frac{1}{2} \sigma_1^2 s^2 \partial_{11} V(s, 1, 1) - (\alpha + \lambda)V(s, 1, 1) & \text{for } s \in (K, \infty) \end{cases}$$

depending on the unknown b_1 subject to the boundary and pasting conditions

$$\begin{aligned} \lim_{s \rightarrow \infty} V(s, 1, 1) &= 0, \\ V(K-, 1, 1) &= V(K+, 1, 1), \\ \partial_1 V(K-, 1, 1) &= \partial_1 V(K+, 1, 1), \\ K - b_1 &= V(b_1+, 1, 1), \\ -1 &= \partial_1 V(b_1+, 1, 1), \end{aligned} \quad (2.2)$$

where $-$ and $+$ indicate taking left and right limits at the corresponding argument, respectively.

The well-known solution of the above equation has the form

$$V(s, 1, 1) = \begin{cases} c_1 s^{\beta^+} + c_2 s^{\beta^-} + h(s) & \text{for } s \in (b_1, K) \\ d_1 s^{\beta^+} + d_2 s^{\beta^-} & \text{for } s \in (K, \infty) \end{cases} \quad (2.3)$$

with unknown coefficients c_1, c_2, d_1, d_2 , and β^+ (β^-) being the positive (negative) root of equation (1.6). For the choice of the function h we refer to Remark 1.7(ii).

Certainly, the first of the conditions under (2.2) implies $d_1 = 0$, and the other four conditions yield

$$\begin{aligned} c_1 K^{\beta^+} + c_2 K^{\beta^-} + h(K) &= d_2 K^{\beta^-}; \\ c_1 \beta^+ K^{\beta^+} + c_2 \beta^- K^{\beta^-} + K h'(K) &= d_2 \beta^- K^{\beta^-}; \\ K - b_1 &= c_1 b_1^{\beta^+} + c_2 b_1^{\beta^-} + h(b_1); \\ -b_1 &= c_1 \beta^+ b_1^{\beta^+} + c_2 \beta^- b_1^{\beta^-} + b_1 h'(b_1). \end{aligned} \quad (2.4)$$

Note that the coefficients c_1, c_2, d_2 linearly depend on $b_1^{\beta^\pm}$ so that the problem comes down to solving numerically for b_1 . For verification we refer to the Appendix.

This concludes the discussion of V for $y = 1$. We now turn to the cases (iii) and (iv) of Theorem 1.6 dealing with the case $y = 0$.

2.2 Proof of Theorem 1.6 (iii)

Recall the setup of Theorem 1.4 but also introduce

$$\tilde{V}_0(s) \stackrel{\text{def}}{=} \sup \left\{ \tilde{\mathbf{E}}_s[e^{-\alpha \tilde{\tau}}(K - \tilde{S}_{\tilde{\tau} \wedge \tilde{\tau}_{s_0}})^+] : \tilde{\tau} \text{ stopping time with respect to } (\tilde{S}_t, t \geq 0) \right\}$$

for all $s \geq s_0$.

Lemma 2.1. *If $b_0 \leq s_0 < K$ then*

$$\tilde{V}_0(s) = (K - s_0) \left(\frac{s}{s_0} \right)^{\gamma^-} \quad \text{for } s \geq s_0, \quad (2.5)$$

and the (possibly infinite) Markov time $\tilde{\tau}_{s_0}$ is the optimal time.

Proof. Assume $b_0 \leq s_0 < K$ and fix $s \geq s_0$. By ‘guess and verify’, it suffices to check that the right-hand side of (2.5) satisfies

$$(v1) \quad (K - s_0)(s/s_0)^{\gamma^-} = \tilde{\mathbf{E}}_s[e^{-\alpha \tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0} < \infty\}}];$$

$$(v2) \quad \text{the process } (e^{-\alpha t}(K - s_0)(\tilde{S}_{t \wedge \tilde{\tau}_{s_0}}/s_0)^{\gamma^-}, t \geq 0) \text{ is a } \tilde{\mathbf{P}}_s\text{-supermartingale;}$$

$$(v3) \quad (K - s_0)(s/s_0)^{\gamma^-} \geq (K - s)^+.$$

For (v1), by Itô's formula,

$$\begin{aligned}
& \tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0} < \infty\}}] \\
&= \lim_{t \rightarrow \infty} \tilde{\mathbf{E}}_s[e^{-\alpha(\tilde{\tau}_{s_0} \wedge t)}(K - s_0) \left(\frac{\tilde{S}_{\tilde{\tau}_{s_0} \wedge t}}{s_0} \right)^{\gamma^-}] \\
&= \lim_{t \rightarrow \infty} \tilde{\mathbf{E}}_s \left[(K - s_0) \left(\frac{s}{s_0} \right)^{\gamma^-} + (K - s_0) \int_0^{\tilde{\tau}_{s_0} \wedge t} e^{-\alpha u} [(\tilde{L} - \alpha I) \left(\frac{\cdot}{s_0} \right)^{\gamma^-}] (\tilde{S}_u) du + M_{t \wedge \tilde{\tau}_{s_0}} \right] \\
&= (K - s_0) \left(\frac{s}{s_0} \right)^{\gamma^-}
\end{aligned}$$

because $(M_t, t \geq 0)$ is a $\tilde{\mathbf{P}}_s$ -martingale and the expression inside the integral vanishes.

For (v2), by Markov Property, it suffices to prove that $\tilde{\mathbf{E}}_s[e^{-\alpha t}(\tilde{S}_{t \wedge \tilde{\tau}_{s_0}}/s_0)^{\gamma^-}] \leq (s/s_0)^{\gamma^-}$ for all $t \geq 0$ ignoring the constant $K - s_0$. But, for fixed $t \geq 0$,

$$\tilde{\mathbf{E}}_s[e^{-\alpha t} \left(\frac{\tilde{S}_{t \wedge \tilde{\tau}_{s_0}}}{s_0} \right)^{\gamma^-}] \leq \tilde{\mathbf{E}}_s[e^{-\alpha(t \wedge \tilde{\tau}_{s_0})} \left(\frac{\tilde{S}_{t \wedge \tilde{\tau}_{s_0}}}{s_0} \right)^{\gamma^-}] = \left(\frac{s}{s_0} \right)^{\gamma^-}.$$

where the last equality was already verified when proving (v1) above.

For (v3), note that

$$\tilde{V}'(b_0) = (K - b_0) \frac{\gamma^-}{b_0} = -1$$

as mentioned in Remark 1.5(i). Based on two arguments, we can now deduce that the derivative of $(K - s_0)(\cdot/s_0)^{\gamma^-}$ is bigger than -1 on $s \in (s_0, K)$. First, the derivative of $(K - s_0)(\cdot/s_0)^{\gamma^-}$ is bounded below by -1 at s_0 because $b_0 \leq s_0$ implies

$$\frac{\tilde{V}'(b_0)}{(K - s_0)\gamma^-/s_0} = \frac{(K - b_0)s_0}{(K - s_0)b_0} \geq 1,$$

and, second, $(K - s_0)(\cdot/s_0)^{\gamma^-}$ is convex on (s_0, K) .

But, if the derivative of $(K - s_0)(\cdot/s_0)^{\gamma^-}$ is bigger than -1 on (s_0, K) and $(K - s_0)(\cdot/s_0)^{\gamma^-}$ touches $(K - \cdot)$ at s_0 , then $(K - s_0)(s/s_0)^{\gamma^-} > (K - s)^+$ for all $s \in (s_0, K)$. Finally, $(K - s_0)(s/s_0)^{\gamma^-} > (K - s)^+ = 0$ for $s \in [K, \infty)$ is obvious. \square

Remark 2.2. In what follows, we are going to use an easy application of Meyer's, [8], generalised Itô's formula which goes as follows: if $\phi : (0, \infty) \rightarrow \mathbb{R}$ is a function which is twice continuously differentiable, except at finitely many points $\{a_1, \dots, a_n\}$, such that

$$\phi'(a_k \pm) \stackrel{\text{def}}{=} \lim_{x \rightarrow a_k \pm} \phi'(x) \quad \text{and} \quad \phi''(a_k \pm) \stackrel{\text{def}}{=} \lim_{x \rightarrow a_k \pm} \phi''(x)$$

exist and are finite, $k = 1, \dots, n$, then³

$$\phi(S_t) = \phi(S_0) + \int_0^t \phi'(S_u) dS_u + \frac{1}{2} \int_0^t \phi''(S_u) d\langle S \rangle_u + \sum_{k=1}^n \frac{1}{2} L_t(a_k) [\phi'(a_k+) - \phi'(a_k-)]$$

³Note that $(S_t, t \geq 0)$ only takes positive values.

for all $t \geq 0$ a.s. where L_t stands for the local time of the continuous semimartingale $(S_t, t \geq 0)$. Note that both integrands in the above formula are well-defined for Lebesgue almost every u almost surely which uniquely determines the integrals.

If ϕ' is continuous then the local time terms would even vanish so that the above formula would look like the classical Itô's formula.

We now return to our problem of finding $V(s, 0, 1)$ for $s > s_0$. Recall that we already know $V(s, 1, i)$ for all $s > 0$ and $i = 0, 1$.

2.2.1 Proof of Theorem 1.6 (iii)(a) and (iii)(b)

Suppose that either condition (a) or condition (b) of Theorem 1.6(iii) is satisfied. Recall $\tau^* = \tau_{b_1} \wedge \tau_{\eta, 0}$ and introduce

$$V^*(s, y, i) \stackrel{\text{def}}{=} \begin{cases} V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} & : s > s_0, y = 0, i = 1, \\ V(s, 1, i) & : s > 0, y = 1, i \in \{0, 1\}. \end{cases}$$

Verifying, for any $s > s_0$,

$$(v1) \quad V^*(s, 0, 1) = \mathbf{E}_{s, 0, 1}[e^{-\alpha\tau^*}(K - S_{\tau^*})^+ \mathbf{1}_{\{\tau^* < \infty\}}],$$

$$(v2) \quad \text{the process } (e^{-\alpha t} V^*(S_t, Y_t, \eta_t), t \geq 0) \text{ is a } \mathbf{P}_{s, 0, 1} \text{-supermartingale,}$$

$$(v3) \quad V^*(s, 0, 1) \geq (K - s)^+,$$

would imply the conclusion of Theorem 1.6 in both cases (a) and (b).

We are going to verify (v1), (v2), (v3). First observe that $V(s_0, 1, 1) > (K - s_0)^+$ holds in both cases (a) and (b). To see this in the non-trivial case (a), note that $(K - b_0)(\cdot/b_0)^{\gamma^-}$ is strictly convex on $[s_0, b_0]$ and touches $(K - s)^+$ at $s = b_0 < K$, so $V(s_0, 1, 1) \geq (K - b_0)(s_0/b_0)^{\gamma^-} > K - s_0$.

As a consequence, $(s_0, 1, 1)$ is in the continuation region with respect to the optimal stopping problem (1.3) on page 4. Since b_1 is the lower boundary of the continuation region when $y = i = 1$ we can deduce that $b_1 < s_0$, and hence, $\tau_{s_0} < \tau^*$ for $s > s_0$. Therefore

$$\begin{aligned} & \mathbf{E}_{s, 0, 1}[e^{-\alpha\tau^*}(K - S_{\tau^*})^+ \mathbf{1}_{\{\tau_{s_0} < \infty\}}] \\ &= \mathbf{E}_{s, 0, 1} \left[\mathbf{E}_{s, 0, 1}[e^{-\alpha\tau^*}(K - S_{\tau^*})^+ \mathbf{1}_{\{\tau_{s_0} < \infty\}} | \mathcal{F}_{\tau_{s_0}}] \right] \\ &= \mathbf{E}_{s, 0, 1} \left[e^{-\alpha\tau_{s_0}} \mathbf{1}_{\{\tau_{s_0} < \infty\}} \underbrace{\mathbf{E}_{s_0, 1, 1}[e^{-r\tau^*}(K - S_{\tau^*})^+]}_{V(s_0, 1, 1)} \right] \end{aligned}$$

by strong Markov property. So, simply working out the Laplace transform of the hitting time τ_{s_0} yields (v1).

Next we verify (v2) for $s > s_0$ fixed. By Markov Property, we only need to show that

$$\mathbf{E}_{s, 0, 1}[e^{-\alpha t} V^*(S_t, Y_t, \eta_t)] \leq V^*(s, 0, 1) \quad (2.6)$$

for all $t \geq 0$.

Fix $t \geq 0$ and consider

$$\begin{aligned} e^{-\alpha t} V^*(S_t, Y_t, \eta_t) &= e^{-\alpha t} V^*(S_t, Y_t, \eta_t) - e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, Y_{t \wedge \tau_{s_0}}, \eta_{t \wedge \tau_{s_0}}) \\ &\quad + e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, 0, 1) \end{aligned} \quad (2.7)$$

where the last term is justified by $V^*(s_0, 1, 1) = V^*(s_0, 0, 1)$.

Now realise that, by strong Markov property,

$$\begin{aligned} &\mathbf{E}_{s,0,1} [\mathbf{1}_{\{t \geq \tau_{s_0}\}} e^{-\alpha t} V^*(S_t, Y_t, \eta_t)] \\ &= \mathbf{E}_{s,0,1} \left[\mathbf{1}_{\{t \geq \tau_{s_0}\}} \mathbf{E}_{s,0,1} [e^{-\alpha t} V^*(S_t, Y_t, \eta_t) | \mathcal{F}_{t \wedge \tau_{s_0}}] \right] \\ &= \int \mathbf{1}_{\{t \geq \tau_{s_0}(\omega)\}} e^{-\alpha \tau_{s_0}(\omega)} \underbrace{\mathbf{E}_{s_0,1,1} [e^{-\alpha(t - \tau_{s_0}(\omega))} V^*(S_{t - \tau_{s_0}(\omega)}, Y_{t - \tau_{s_0}(\omega)}, \eta_{t - \tau_{s_0}(\omega)})]}_{\leq V^*(s_0, 1, 1) \text{ from case (ii)}} \mathbf{P}_{s,0,1}(d\omega), \end{aligned}$$

and, since the difference on the right-hand side of (2.7) equals

$$\left[e^{-\alpha t} V^*(S_t, Y_t, \eta_t) - e^{-\alpha \tau_{s_0}} V^*(s_0, 1, 1) \right] \times \mathbf{1}_{\{t \geq \tau_{s_0}\}},$$

we obtain that

$$\mathbf{E}_{s,0,1} [e^{-\alpha t} V^*(S_t, Y_t, \eta_t)] \leq \mathbf{E}_{s,0,1} [e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, 0, 1)]. \quad (2.8)$$

To prove (2.6), we want to apply Itô's formula on the above right-hand side followed by taking expectations.

Recall the operator L introduced on page 3. As the function $V^*(\cdot, 0, 1)$ defined on (s_0, ∞) can be extended to a C^2 -function on \mathbb{R} , Itô's formula $\mathbf{P}_{s,0,1}$ -a.s. yields

$$e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, 0, 1) = V^*(s, 0, 1) + \int_0^{t \wedge \tau_{s_0}} e^{-\alpha u} (L - \alpha I) V^*(S_u, 0, 1) du + I_{BM}$$

where I_{BM} is an integrable stochastic integral against Brownian motion whose expectation vanishes.

Furthermore, by explicit calculation, $(L - \alpha I) V^*(s', 0, 1) = 0$ for $s' > s_0$, and hence the right-hand side of (2.8) reduces to $V^*(s, 0, 1)$ eventually showing (2.6).

It remains to verify (v3) for any $s > s_0$ in both cases (a) and (b).

For (a), observe that

$$V^*(s, 0, 1) = V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} \geq (K - b_0) \left(\frac{s_0}{b_0} \right)^{\gamma^-} \left(\frac{s}{s_0} \right)^{\gamma^-} = (K - b_0) \left(\frac{s}{b_0} \right)^{\gamma^-},$$

and hence it suffices to show that $(K - b_0)(s/b_0)^{\gamma^-} \geq (K - s_0)^+$ for $s > s_0$. Note that $(K - b_0)(\cdot/b_0)^{\gamma^-}$ coincides on $[b_0, \infty)$ with the value function \tilde{V} given in Theorem 1.4 which satisfies both $\tilde{V}(s) \geq (K - s)^+$ for $s \in [b_0, \infty)$ and $\tilde{V}'(b_0) = -1$. Therefore, because $(K - b_0)(\cdot/b_0)^{\gamma^-}$ is a convex function on $(0, \infty)$, it must be bounded below by $(K - \cdot)^+$ on (s_0, b_0) , too.

For (b), there is nothing to show if $s_0 \geq K$. But, if $b_0 \leq s_0 < K$, then we know from the proof of Lemma 2.1 that $(K - s_0)(s/s_0)^{\gamma^-} > (K - s)^+$ for all $s > s_0$. Thus

$$V^*(s, 0, 1) = V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} > (K - s)^+ \quad \text{for } s > s_0$$

because $V(s_0, 1, 1) > (K - s_0)^+$ by assumption in case (b).

2.2.2 Proof of Theorem 1.6 (iii)(c)

We are going to verify (v1),(v2),(v3) for fixed $s > s_0$ as in Section 2.2.1 but using $\tau^* = \tau_{s_0}$.

First, note that, on $\{\tau_{s_0} < \infty\}$, the process $(S_t, 0 \leq t \leq \tau_{s_0})$ under $\mathbf{P}_{s,0,1}$ has the same distribution as the process $(\tilde{S}_t, 0 \leq t \leq \tilde{\tau}_{s_0})$ under $\tilde{\mathbf{P}}_s$ introduced in Theorem 1.4. By Lemma 2.1, for $s > s_0$, we therefore have

$$\begin{aligned} & \mathbf{E}_{s,0,1}[e^{-\alpha\tau_{s_0}}(K - S_{\tau_{s_0}})^+ \mathbf{1}_{\{\tau_{s_0} < \infty\}}] \\ &= \tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0} < \infty\}}] = (K - s_0) \left(\frac{s}{s_0}\right)^{\gamma^-} \end{aligned}$$

where $V(s_0, 1, 1) = K - s_0$ by assumption showing (v1).

For (v2), one can copy the corresponding proof in Section 2.2.1 because the value function has the same form in all sub-cases (a,b,c).

Finally, $V^*(s, 0, 1) = \tilde{V}_0(s) \geq (K - s)^+$ for $s > s_0$, showing (v3) and finishing the proof of Theorem 1.6(iii).

2.3 Proof of Theorem 1.6 (iv)

As in the proof of Theorem 1.6(ii), we first guess the structure of the stopping region and then verify that the solution of the corresponding free-boundary value problem is the wanted value function.

First, as the value function is claimed to be attained at the Markov time $\tau_{[b_*, b_0], 0} \wedge \tau_{b_1} \wedge \tau_{\eta, 0}$, the stopping region to be guessed should take the form

$$\left[[b_*, b_0] \times \{0\} \times \{1\} \right] \cup \left[(0, b_1] \times \{1\} \times \{1\} \right] \cup \left[(0, \infty) \times \{1\} \times \{0\} \right]$$

where b_0, b_1 are already known but $b_* \in [s_0, b_0)$ is not. Recall that both b_0 and b_1 must be less than K .

Now, referring to the proof of Theorem 1.6(ii) for the underlying argument, the corresponding free-boundary value problem for the unknown value function $V(s, 0, 1)$ is

$$0 = \mu_0 s \partial_1 V(s, 0, 1) + \frac{1}{2} \sigma_0^2 s^2 \partial_{11} V(s, 0, 1) - \alpha V(s, 0, 1) \quad \text{for } s \in (s_0, b_*) \cup (b_0, \infty)$$

depending on the unknown b_* subject to the boundary conditions

$$\begin{aligned} V(s_0, 1, 1) &= V(s_0+, 0, 1); \\ V(b_*-, 0, 1) &= K - b_*; \\ \partial_1 V(b_*-, 0, 1) &= -1; \\ K - b_0 &= V(b_0+, 0, 1); \\ -1 &= \partial_1 V(b_0+, 0, 1); \\ \lim_{s \rightarrow \infty} V(s, 0, 1) &= 0. \end{aligned} \tag{2.9}$$

Taking into account Theorem 1.4, if $V(\cdot, 0, 1)$ satisfies these constraints then it must have the

representation

$$\begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : s \in (s_0, b_*), \\ K - s & : s \in [b_*, b_0] \cap (s_0, \infty), \\ (K - b_0) \left(\frac{s}{b_0} \right)^{\gamma^-} & : s \in (b_0, \infty), \end{cases} \quad (2.10)$$

where e_1^* , e_2^* , b_* should be determined by the first three conditions of (2.9). However, since V is the value function associated with an optimal stopping problem, there is a fourth constraint on the choice of e_1^* , e_2^* , b_* which is used in the next lemma.

Lemma 2.3. *If $s_0 < b_0$ and $(K - s_0) < V(s_0, 1, 1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, then there exist unique coefficients e_1^* , e_2^* and a unique stopping level $b_* \in (s_0, b_0)$ such that*

$$V(s_0, 1, 1) = e_1^* s_0^{\gamma^+} + e_2^* s_0^{\gamma^-}; \quad (2.11)$$

$$e_1^* b_*^{\gamma^+} + e_2^* b_*^{\gamma^-} = K - b_*; \quad (2.12)$$

$$e_1^* \gamma^+ b_*^{\gamma^+} + e_2^* \gamma^- b_*^{\gamma^-} = -b_*;$$

$$e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} > K - s \quad \text{for } s \in (s_0, b_*).$$

Before proving this lemma, we are going to state the following preparatory results.

Lemma 2.4. *(see [6, Lemma 2] for example) For fixed $s_0 \leq s \leq \tilde{s}$,*

$$\phi_1(s, \tilde{s}) \stackrel{\text{def}}{=} \mathbf{E}_{s,0,1}[e^{-\alpha(\tau_{s_0} \wedge \tau_{\tilde{s}})} \mathbf{1}_{\{\tau_{s_0} < \tau_{\tilde{s}}\}}] = \frac{s^{\gamma^+} \tilde{s}^{\gamma^-} - s^{\gamma^-} \tilde{s}^{\gamma^+}}{s_0^{\gamma^+} \tilde{s}^{\gamma^-} - s_0^{\gamma^-} \tilde{s}^{\gamma^+}}$$

and

$$\phi_2(s, \tilde{s}) \stackrel{\text{def}}{=} \mathbf{E}_{s,0,1}[e^{-\alpha(\tau_{s_0} \wedge \tau_{\tilde{s}})} \mathbf{1}_{\{\tau_{s_0} > \tau_{\tilde{s}}\}}] = \frac{s_0^{\gamma^+} s^{\gamma^-} - s_0^{\gamma^-} s^{\gamma^+}}{s_0^{\gamma^+} \tilde{s}^{\gamma^-} - s_0^{\gamma^-} \tilde{s}^{\gamma^+}}$$

where

$$\tau_{\tilde{s}} \stackrel{\text{def}}{=} \inf\{t \geq 0 : S_t \geq \tilde{s}\}.$$

Lemma 2.5. *Under the assumptions of Lemma 2.3, the equation*

$$V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} = K - s$$

has exactly two solutions $s_1, s_2 \in (s_0, K)$ only one of which, say s_1 , is less than b_0 .

Proof. Consider the function $f(\cdot) = V(s_0, 1, 1)(\cdot/s_0)^{\gamma^-} - (K - \cdot)$ on $(0, \infty)$, and note that $f(s_0) > 0$, $f(b_0) < 0$, $f(K) > 0$. By Intermediate Value Theorem, there exist $s_0 < s_1 < b_0$ and $b_0 < s_2 < K$ such that $f(s_1) = f(s_2) = 0$, that is,

$$V(s_0, 1, 1) \left(\frac{s_i}{s_0} \right)^{\gamma^-} = K - s_i \quad \text{for } i = 1, 2.$$

There exist exactly two points, only, because $V(s_0, 1, 1)(\cdot/s_0)^{\gamma^-}$ is strictly convex on $(0, \infty)$ and can be intersected by a line at no more than two points. Moreover,

$$V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} < K - s \quad \text{for} \quad s_1 < s < s_2, \quad (2.13)$$

because of strict convexity, too. \square

Proof of Lemma 2.3. Introduce

$$\Gamma(s, \tilde{s}) \stackrel{\text{def}}{=} V(s_0, 1, 1)\phi_1(s, \tilde{s}) + (K - \tilde{s})\phi_2(s, \tilde{s}) \quad \text{for} \quad s_0 \leq s \leq \tilde{s}$$

using the functions ϕ_1, ϕ_2 defined in Lemma 2.4. For fixed $\tilde{s} > s_0$, note that the function $[s_0, \tilde{s}] \ni s \mapsto \Gamma(s, \tilde{s})$ is of the form $e_1 s^{\gamma^+} + e_2 s^{\gamma^-}$ with

$$e_1 = \frac{V(s_0, 1, 1)\tilde{s}^{\gamma^-} - (K - \tilde{s})s_0^{\gamma^-}}{s_0^{\gamma^+}\tilde{s}^{\gamma^-} - s_0^{\gamma^-}\tilde{s}^{\gamma^+}}, \quad e_2 = \frac{-V(s_0, 1, 1)\tilde{s}^{\gamma^+} + (K - \tilde{s})s_0^{\gamma^+}}{s_0^{\gamma^+}\tilde{s}^{\gamma^-} - s_0^{\gamma^-}\tilde{s}^{\gamma^+}},$$

and that both boundary conditions

$$e_1 s_0^{\gamma^+} + e_2 s_0^{\gamma^-} = \Gamma(s_0, \tilde{s}) = V(s_0, 1, 1), \quad e_1 \tilde{s}^{\gamma^+} + e_2 \tilde{s}^{\gamma^-} = \Gamma(\tilde{s}, \tilde{s}) = K - \tilde{s}$$

are satisfied. Hence, if we can show that there is exactly one $b_* \in (s_0, b_0)$ such that both

$$\partial_1 \Gamma(b_*, b_*) = -1 \quad \text{and} \quad \Gamma(s, b_*) > K - s \quad \text{for} \quad s \in (s_0, b_*),$$

then the triplet (e_1^*, e_2^*, b_*) , where e_1^*, e_2^* are given by the above formulae for e_1, e_2 when replacing \tilde{s} by b_* , would be the unique solution of the problem stated in Lemma 2.3. Here the uniqueness of e_1^*, e_2^* follows from the uniqueness of b_* as the formulae for e_1^*, e_2^* coincide with the unique solution to the sub-system (2.11),(2.12) of the conditions in Lemma 2.3 when treating $s_0^{\gamma^\pm}$ and $b_*^{\gamma^\pm}$ as coefficients.

First, we show that there is $b_* \in (s_0, b_0)$ such that $\partial_1 \Gamma(b_*, b_*) = -1$. For the uniqueness of b_* we refer to Remark 2.6(ii) below.

Using simple calculations based on Itô's formula, observe that, for any $s \geq s_0$, the stochastic process $(e^{-\alpha(t \wedge \tau_{s_0})} V(s_0, 1, 1)(S_{t \wedge \tau_{s_0}}/s_0)^{\gamma^-}, t \geq 0)$ is a $\mathbf{P}_{s_0, 1}$ -martingale. Now recall s_1 from Lemma 2.5 and the stochastic representation of $\Gamma(s, \tilde{s})$ in terms of ϕ_1, ϕ_2 . Then, by Doob's Optional Sampling Theorem, for $s_0 \leq s \leq s_1$,

$$\begin{aligned} V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} &= \mathbf{E}_{s_0, 1}[e^{-\alpha(\tau_{s_0} \wedge \tau^{s_1})} V(s_0, 1, 1)\left(\frac{S_{\tau_{s_0} \wedge \tau^{s_1}}}{s_0}\right)^{\gamma^-}] \\ &= V(s_0, 1, 1)\phi_1(s, s_1) + V(s_0, 1, 1)\left(\frac{s_1}{s_0}\right)^{\gamma^-} \phi_2(s, s_1) \\ &= V(s_0, 1, 1)\phi_1(s, s_1) + (K - s_1)\phi_2(s, s_1) \\ &= \Gamma(s, s_1) \end{aligned}$$

using Lemma 2.5 to justify the penultimate equality above. Thus

$$\partial_1 \Gamma(s_1, s_1) = \partial_s \left[V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} \right]_{s=s_1}. \quad (2.14)$$

Next, for any $s > s_0$,

$$\partial_1 \Gamma(s, s) = \frac{V s^{\gamma^+ + \gamma^- - 1} (\gamma^+ - \gamma^-) + (K - s) s^{-1} (\gamma^- s_0^{\gamma^+} s^{\gamma^-} - \gamma^+ s_0^{\gamma^-} s^{\gamma^+})}{s_0^{\gamma^+} s^{\gamma^-} - s_0^{\gamma^-} s^{\gamma^+}}$$

where V stands for $V(s_0, 1, 1)$. Consider the above right-hand side as a function of (V, s) which we denote by $g(V, s)$ in what follows.

Choose $s = b_0$ and realise that the function $V \mapsto g(V, b_0)$ is strictly decreasing since $s_0 < b_0$ implies $s_0^{\gamma^+} b_0^{\gamma^-} - s_0^{\gamma^-} b_0^{\gamma^+} < 0$. Moreover $g((K - b_0)(s_0/b_0)^{\gamma^-}, b_0) = -1$, so that our assumption of $V(s_0, 1, 1) < (K - b_0)(s_0/b_0)^{\gamma^-}$ implies $g(V(s_0, 1, 1), b_0) > -1$.

But, using (2.14), we also have that

$$g(V(s_0, 1, 1), s_1) = \partial_s \left[V(s_0, 1, 1) \left(\frac{s}{s_0} \right)^{\gamma^-} \right]_{s=s_1},$$

where, by the same arguments used in the proof of Lemma 2.5, the above right-hand side must be less than -1 .

All in all, we obtain that $g(V(s_0, 1, 1), s_1) < -1 < g(V(s_0, 1, 1), b_0)$. And since the function $g(V(s_0, 1, 1), \cdot)$ is continuous on (s_0, ∞) , and since $s_0 < s_1 < b_0$ by Lemma 2.5, it is again a consequence of the Intermediate Value Theorem that there exists $b_* \in (s_1, b_0)$ such that $\partial_1 \Gamma(b_*, b_*) = g(V(s_0, 1, 1), b_*) = -1$.

Second, finishing the proof of Lemma 2.3, we will show that

$$\Gamma(s, b_*) = e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} > K - s \quad \text{for } s \in (s_0, b_*)$$

where

$$e_1^* = \frac{V(s_0, 1, 1) b_*^{\gamma^-} - (K - b_*) s_0^{\gamma^-}}{s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}}, \quad e_2^* = \frac{-V(s_0, 1, 1) b_*^{\gamma^+} + (K - b_*) s_0^{\gamma^+}}{s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}}.$$

In order to do so we analyse the function $f(s) = e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-}$, $s > 0$. Since $f'(b_*) = -1$ has already been shown, $f(s) > K - s$ for $s \in (s_0, b_*)$ would follow from f being strictly convex on $(0, b_*)$ which we are going to prove below.

First observe that both e_1^* and e_2^* are positive. In fact, as $s_0 < b_*$ implies that the denominator $s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}$ is negative, the positivity of the two coefficients e_1^*, e_2^* follows from

$$V(s_0, 1, 1) b_*^{\gamma^-} - (K - b_*) s_0^{\gamma^-} < 0 \quad \text{and} \quad -V(s_0, 1, 1) b_*^{\gamma^+} + (K - b_*) s_0^{\gamma^+} < 0$$

where the former inequality is a consequence of (2.13) because b_* was chosen from the interval (s_1, b_0) while the latter inequality is a consequence of our assumption $K - s_0 < V(s_0, 1, 1)$, on the one hand, and $s_0 < b_*, \gamma^+ > 0$, on the other.

As a consequence, if $\gamma^+ \geq 1$, then f is the sum of two strictly convex functions, and hence strictly convex everywhere.

Now assume $0 < \gamma^+ < 1$ which is the remaining case. Note that f has exactly one local minimum at $s' = \left[\frac{-e_2^* \gamma^-}{e_1^* \gamma^+} \right]^{\frac{1}{\gamma^+ - \gamma^-}}$ and that this minimum is global because $f(0+) = \lim_{s \rightarrow \infty} f(s) = \infty$.

Thus, f is strictly decreasing on $(0, s')$ and strictly increasing on (s', ∞) which implies $b_* < s'$ because $f'(b_*) = -1$.

Finally, f has exactly one point of inflection at $s'' = [\frac{-e_2^* \gamma^-(\gamma^- - 1)}{e_1^* \gamma^+(\gamma^+ - 1)}]^{\frac{1}{\gamma^+ - \gamma^-}}$, and this point satisfies $s'' > s'$. Hence, f must be at least strictly convex on $(0, s')$ which also proves its strict convexity on $(0, b_*) \subseteq (0, s')$. \square

Remark 2.6. (i) In the case of $K - s_0 = V(s_0, 1, 1)$, the choice of b_* has not been discussed yet. In this case, we claim that $b_* = s_0$, and we will verify below that the corresponding function given by (2.10) coincides with $V(\cdot, 0, 1)$ on (s_0, ∞) .

(ii) In the case of $(K - s_0) < V(s_0, 1, 1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, we showed existence of $b_* \in (s_0, b_0)$, and any such b_* uniquely determines a function as given by (2.10). We will verify below that any such function coincides with $V(\cdot, 0, 1)$ on (s_0, ∞) . Moreover, as shown in the proof of Lemma 2.3, for any choice of b_* , the corresponding function given by (2.10) must be strictly convex on (s_0, b_*) . Hence, as the value function with respect to an optimal stopping problem is unique, there can only be one b_* .

Set $\tau^* = \tau_{[b_*, b_0], 0} \wedge \tau_{b_1} \wedge \tau_{\eta, 0}$ and introduce

$$V^*(s, y, i) \stackrel{\text{def}}{=} \begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : s \in (s_0, b_*), y = 0, i = 1; \\ K - s & : s \in [b_*, b_0] \cap (s_0, \infty), y = 0, i = 1; \\ (K - b_0) \left(\frac{s}{b_0}\right)^{\gamma^-} & : s \in (b_0, \infty), y = 0, i = 1; \\ V(s, 1, i) & : s > 0, y = 1, i \in \{0, 1\}. \end{cases}$$

Again, by verifying the conditions (v1), (v2), (v3) stated at the beginning of Section 2.2.1 for any fixed $s > s_0$, we complete both the program set out in Remark 2.6 and the proof of Theorem 1.6(iv).

For (v1), if $s \in [b_*, b_0]$, then there is nothing to prove as one stops immediately, and if $s > b_0$, then (v1) follows from Theorem 1.4 as $V^*(\cdot, 0, 1)$ coincides with \tilde{V} on (b_0, ∞) .

In the remaining case of $s_0 < s < b_*$, observe that the assumptions of Lemma 2.3 are satisfied as $K - s_0 = V(s_0, 1, 1)$ can be ruled out. Furthermore,

$$\begin{aligned} & \mathbf{E}_{s, 0, 1} [e^{-\alpha \tau_{[b_*, b_0], 0} \wedge \tau_{b_1} \wedge \tau_{\eta, 0}} (K - S_{\tau_{[b_*, b_0], 0} \wedge \tau_{b_1} \wedge \tau_{\eta, 0}})^+] \\ &= \mathbf{E}_{s, 0, 1} [e^{-\alpha(\tau_{b_1} \wedge \tau_{\eta, 0})} (K - S_{\tau_{b_1} \wedge \tau_{\eta, 0}})^+ \mathbf{1}_{\{\tau_{[b_*, b_0], 0} > \tau_{s_0}\}}] + \mathbf{E}_{s, 0, 1} [e^{-\alpha \tau_{b_*}} (K - b_*)^+ \mathbf{1}_{\{\tau_{[b_*, b_0], 0} < \tau_{s_0}\}}] \\ &= \mathbf{E}_{s, 0, 1} \left[\mathbf{E}_{s, 0, 1} [e^{-\alpha(\tau_{b_1} \wedge \tau_{\eta, 0})} (K - S_{\tau_{b_1} \wedge \tau_{\eta, 0}})^+ \mathbf{1}_{\{\tau_{[b_*, b_0], 0} > \tau_{s_0}\}} | \mathcal{F}_{\tau_{s_0}}] \right] + (K - b_*) \phi_2(s, b_*) \end{aligned}$$

which, by strong Markov property, simplifies to

$$V(s_0, 1, 1) \phi_1(s, b_*) + (K - b_*) \phi_2(s, b_*).$$

Using the definition of Γ given at the beginning of the proof of Lemma 2.3 on page 14, the last expression equals $\Gamma(s, b_*)$. And since b_* was chosen such that $\Gamma(s, b_*) = V^*(s, 0, 1)$, condition (v1) follows in the remaining case, too.

For (v2), realise that, by the same arguments used in Section 2.2.1 on page 12, the above candidate value function of this section satisfies (2.8), and hence we only need to show that

$$\mathbf{E}_{s, 0, 1} [e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, 0, 1)] \leq V^*(s, 0, 1) \quad (2.15)$$

for any $t \geq 0$.

Now, consider the function $\phi(s') = V^*(s', 0, 1)$, $s' > s_0$. Note that ϕ can be extended to a function on $(0, \infty)$ which is of the type described in Remark 2.2 having finitely many exceptional points of insufficient smoothness. In the case of $K - s_0 = V(s_0, 1, 1)$, where $b_* = s_0$, there is one exceptional point at b_0 , whereas in the case of $K - s_0 < V(s_0, 1, 1)$ there are two exceptional points at b_*, b_0 . However, in both cases, applying Theorem 1.4 and Lemma 2.3, respectively, one can extend ϕ in such a way that ϕ' is continuous.

Thus, according to Remark 2.2, we $\mathbf{P}_{s,0,1}$ -a.s. have

$$e^{-\alpha(t \wedge \tau_{s_0})} V^*(S_{t \wedge \tau_{s_0}}, 0, 1) = V^*(s, 0, 1) + \int_0^{t \wedge \tau_{s_0}} e^{-\alpha u} (L - \alpha I) V^*(S_u, 0, 1) du + I_{BM} \quad (2.16)$$

where L stands once more for the operator introduced on page 3, and I_{BM} is an integrable stochastic integral against Brownian motion whose expectation vanishes.

Next, by explicit calculation, $(L - \alpha I) V^*(\cdot, 0, 1) \leq 0$ on $(s_0, b_*) \cup (b_*, b_0) \cup (b_0, \infty)$. Since the process $(S_u, u \in [0, \tau_{s_0}])$ has $\mathbf{P}_{s,0,1}$ -a.s. no occupation time in $\{b_*, b_0\}$, inequality (2.15) follows from (2.16) by taking expectations, proving (v2).

Finally, condition (v3) is a consequence of Lemma 2.3, if $s \in (s_0, b_*)$, and of Theorem 1.4, if $s \in (b_0, \infty)$. Otherwise, there is nothing to be shown.

3 Numerical Analysis and Discussion

We are going to discuss the four cases (iii)(a-c) and (iv) of Theorem 1.6 using practically relevant values for $s_0, \mu_0, \sigma_0, \mu_1, \sigma_1, \alpha, \lambda, K$.

Note that the choice of $\mu_1, \sigma_1, \lambda, K$ fixes the value of b_1 and that the two cases (iii)(b,c) of Theorem 1.6 can be reformulated as

(b') $b_0 \leq s_0$ and $b_1 < s_0$;

(c') $b_0 \leq s_0 < K$ and $b_1 \geq s_0$.

However, the formulation of the two cases (iii)(a) and (iv) requires the value of $V(s_0, 1, 1)$, and that's why we decided to formulate (iii)(b,c) using $V(s_0, 1, 1)$, too.

In what follows, when using the noun 'put' without further specification, we mean a *perpetual* American put as considered in Theorem 1.6. However, as motivated in Remark 1.1(ii,iii,v), the average length of the put's optimal exercise time is supposed to be rather short, and hence we think that our analysis also produces good benchmarks for traded American puts with times to maturity being long enough to allow for medium term option trading, that is, three months and longer.

First, we have to choose the put's underlying asset. By the macroeconomic explanation given by Black in [1], we think that a leverage effect is more likely to be observed when the whole market falls, and hence we choose an index, say, the Dow Jones index.

Second, to fully determine the put, a strike level has to be chosen. At the end of this section we give a summary of how to choose the strike level motivated by our discussion of Theorem 1.6 below. For now we choose $K = 17000$ for demonstration.

Next we fix the following hypothetical values for $\sigma_0 = 20\%$, $\mu_1 = 0$, $\sigma_1 = 35\%$, $\alpha = 5\%$, $\lambda = 100$ and refer to Remark 1.1 and Remark 1.3(i) for their interpretation.

So, $\sigma_0 = 20\%$ is supposed to be the implied volatility at present time of a traded American put with strike K and time to maturity of at least three months, and we assume that the expected market drop would cause an ‘excited’ volatility of $\sigma_1 = 35\%$.

Setting $\lambda = 100$ means to assume that the ‘excited’ state would only last for half a week on average.

After the market has dropped, it is not clear what the new trend μ_1 of the index would be. Furthermore, if the ‘excited’ state only lasts for a short period of time, one can assume that the ‘excited’ fluctuations according to the bigger σ_1 dominate the trend. Thus a reasonable choice for the new trend would be $\mu_1 = 0$.

The two remaining non-fixed parameters are s_0 and μ_0 . Since σ_0, α, K have been fixed, there is a one-to-one correspondence between μ_0 and b_0 , and hence, each pair (μ_0, s_0) determines one of the four cases (iii)(a-c) and (iv) of Theorem 1.6. The optimal stopping rules given in each of these cases are called *strategies* of the trader, in what follows.

In practice, depending on the present value s of the Dow Jones, the trader would choose s_0 according to their preferences of the future—they expect a market drop of a certain size. In our analysis we take the reverse point of view: we first classify the values of s_0 and then discuss the impact of present values s above s_0 on the strategy to be chosen by the trader.

While $s - s_0$ determines the size of the expected market drop, the choice of μ_0 determines how soon this is supposed to happen in terms of the model (recall that σ_0 has been fixed). A rather small value of μ_0 should be used if one wants that many price-trajectories predicted by the model reach the level s_0 in a rather short time. For example, a value of $\mu_0 = -100\%$ would imply that, roughly, the value of the index expected under the model drops from 15600 to 15000 within two weeks.

In contrast, in the case of bigger values of μ_0 , the model more often predicts rising values of the index in the future, and this is of course not in accordance with an expected market drop. We will nevertheless analyse bigger values of μ_0 because the corresponding strategies might be of use for the trader in case they learn during the trade that their preferences of the future were wrong.

Figure 1 below shows the blue graph of $b_0 = b_0(\mu_0)$ embedded into the (μ_0, s_0) -plane. The red horizontal line marks the level $b_1 = 14658$ which crosses $b_0(\mu_0)$ at $\mu_0 = 13.7\%$.

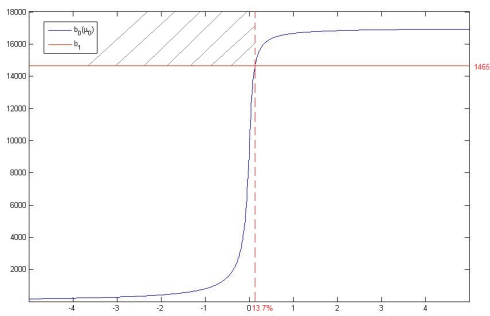


Figure 1

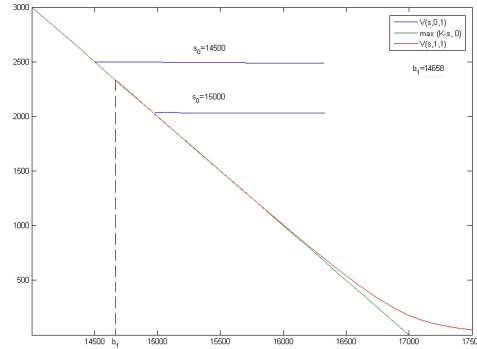


Figure 2

Any point (μ_0, s_0) left (or above) of the curve $b_0(\mu_0)$ is associated with one of the cases (iii)(b,c), and any point right (or below) the curve is associated with one of the cases (iii)(a),(iv) of Theorem 1.6.

Figure 2 shows the value functions corresponding to the two points $(-1, 14500)$ and $(-1, 15000)$ which are both left of the curve $b_0(\mu_0)$ in Figure 1. The green anti-diagonal line is part of the gain function $(K - \cdot)^+$, and the red convex curve is the graph of $V(\cdot, 1, 1)$ which merges onto the gain

function at b_1 . The blue branches hitting $V(\cdot, 1, 1)$ at $s_0 = 14500$ and $s_0 = 15000$, respectively, are the graphs of the corresponding version of $V(\cdot, 0, 1)$ before the regime change at s_0 .

Recall that $V(\cdot, 0, 1)$ and $V(\cdot, 1, 1)$ are two different components of the value function, and they only meet continuously in the above picture because of the boundary condition explained in Remark 1.2(i) on page 4.

Figure 2 can be used to illustrate the qualitative difference between the strategy assuming $s_0 = 14500 \leq b_1$ (case (iii)(c)) and the one assuming $s_0 = 15000 > b_1$ (case (iii)(b)). When assuming $s_0 = 14500$, the trader waits for the index to reach s_0 and would then sell/exercise the put immediately. When assuming $s_0 = 15000$, they would also wait for the index to reach s_0 but would then exploit the regime change from σ_0 to σ_1 implemented into their model due to an implied leverage effect during a market fall: they would either sell/exercise the put after a further waiting time of the order of $1/\lambda$, or sell/exercise the put when the index reaches b_1 . Note that, in our example, $1/\lambda$ equals $1/2$ week which is very short. As a consequence, $V(\cdot, 1, 1)$ looks very similar to how the value function of a traded American put shortly before maturity would look like, and this explains why $V(\cdot, 1, 1)$ is so close to the gain function.

- Remark 3.1.** (i) According to our definition of the value function, all strategies refer to exercising the option. However, since the price of an option which has not matured yet always tops its exercise value, selling the option would not cause any disadvantage.
- (ii) As argued above, choosing a put with strike K such that the level s_0 defining the trade is below b_1 and then applying Theorem 1.6 using a small value of μ_0 , that is, using a value of μ_0 in accordance with an expected market drop results in an optimal strategy where the trader would NOT benefit from the implied leverage effect. So, the trader would want to choose K such that the level s_0 they have in mind is above b_1 .
- (iii) Following (ii), the strategy to be used would be the one described above with respect to $s_0 = 15000$ (case (iii)(b)) for at least all (μ_0, s_0) in the marked area shown in Figure 1. Notice that this area covers values of μ_0 as large as 13.7% which is, for example, well-above the 10 years (2004-2013) average return rate of 6.05% of the Dow Jones. Thus, the strategy given in case (iii)(b) of Theorem 1.6 is robust in the sense that it applies to small values of μ_0 , when the model would predict a market drop in accordance with the preferences of the trader, but also to ‘neutral’ values of μ_0 , when the model would predict standard returns rather than a market drop.

Because of Remark 3.1(ii), we restrict the remaining part of our discussion to cases where $s_0 > b_1$. By Remark 3.1(iii), we know that the strategy given in case (iii)(b) is robust for small and ‘neutral’ values of μ_0 . Next, we discuss the type of strategy offered by Theorem 1.6 when the trader’s preferences for the future are ‘entirely’ wrong, that is, when μ_0 is significantly bigger than 13.7% and (μ_0, s_0) belongs to the quadrant on the right-hand side of the marked area in Figure 1. For demonstration, we choose $\mu_0 = 30\%$.

Figure 3 shows the part of the quadrant on the right-hand side of the marked area in Figure 1 which refers to $10\% \leq \mu_0 \leq 100\%$. The blue upper concave curve is the graph of $b_0(\mu_0)$, and the green concave curve beneath, which meets the upper curve at $\mu_0 = 13.7\%$, is the graph of a function we call $s_0^{max} = s_0^{max}(\mu_0)$. This function gives the root of the equation

$$V(s_0, 1, 1) = (K - b_0) \left(\frac{s_0}{b_0} \right)^{\gamma^-}, \quad s_0 \text{ unknown},$$

which is the value of s_0 at which the switch between case (iii)(a) and case (iv) of Theorem 1.6 occurs. The red horizontal line again marks the level of $b_1 = 14658$, and the black vertical fat bar marks the values of s_0 between b_1 and $s_0^{max} = 15742$ at $\mu_0 = 30\%$.

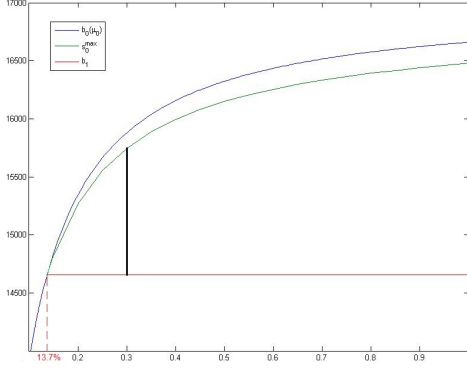


Figure 3

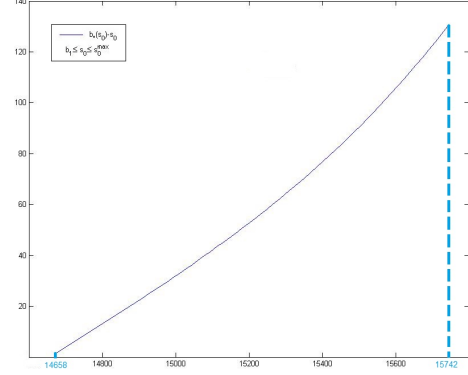


Figure 4

Any point (μ_0, s_0) between the horizontal line and the curve $s_0^{max}(\mu_0)$ is associated with case (iv) of Theorem 1.6, while any point between the two curves $s_0^{max}(\mu_0)$ and $b_0(\mu_0)$ is associated with case (iii)(a).

In case (iv), there exists a corresponding $b_* = b_*(\mu_0, s_0) \in (s_0, b_0)$. For fixed $\mu_0 = 30\%$, we write $b_*(s_0)$ for $b_*(0.3, s_0)$, and Figure 4 shows the graph of $b_*(s_0) - s_0$ for those values of s_0 marked by the vertical fat bar in Figure 3. Note that a further look at the proof of Lemma 2.3 reveals $\lim_{s_0 \uparrow s_0^{max}} b_*(s_0) = b_0$.

Figure 5 below shows the value function corresponding to the point $(0.3, 15000)$ which is a point on the vertical fat bar in Figure 3. To better illustrate the typical shape of the components of this value function, we scaled the axes in a non-linear way which is why, in contrast to the other figures, there are no numerical values assigned to the axes.

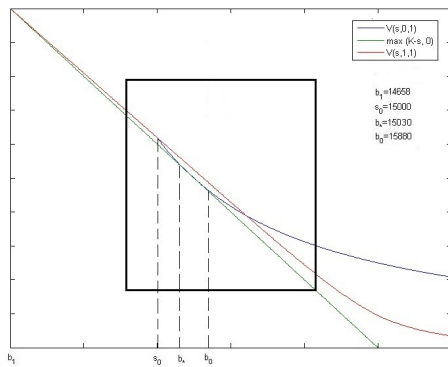


Figure 5

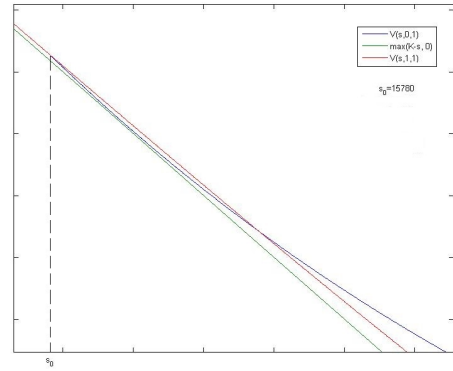


Figure 6

The green anti-diagonal line is part of the gain function $(K - \cdot)^+$, and the red convex curve is the graph of $V(\cdot, 1, 1)$ which merges onto the gain function at b_1 in the upper left corner. The component $V(\cdot, 0, 1)$, plotted in blue, is identical to the gain function between b_* and b_0 but also has two branches: the left branch below $V(\cdot, 1, 1)$ connects $V(\cdot, 1, 1)$ at s_0 with the gain function at b_* , and the right branch crosses $V(\cdot, 1, 1)$ before merging onto the gain function at b_0 .

Figure 6 zooms into the window marked in Figure 5 showing the components of the value function corresponding to the point $(0.3, 15780)$ which is a case-(iii)(a)-point above the vertical fat bar in Figure 3 but still below the curve $b_0(\mu_0)$. Only the component $V(\cdot, 0, 1)$ changes. While, in Figure 5, $V(\cdot, 0, 1)$ is identical to the gain function on a whole interval (b_*, b_0) , in Figure 6, its graph stays above the gain function everywhere crossing $V(\cdot, 1, 1)$ from the right and meeting it again further left at s_0 .

Figure 6 graphically confirms Theorem 1.6 in asserting that the case-(iii)(a)-strategy is identical to the case-(iii)(b)-strategy discussed in the context of Figure 2. All in all, the case-(iii)(b)-strategy would be applicable for all (μ_0, s_0) in both the marked area shown in Figure 1 and in that part of the quadrant on the right-hand side of this marked area which is above the curve $s_0^{max}(\mu_0)$ in Figure 3.

Remark 3.2. Recall that the case-(iii)(b)-strategy involves waiting for the index to fall $s - s_0$ points where, by Remark 1.1(ii), the size of $s - s_0$ is considerable. Thus, for values of μ_0 as large as 30% in our example, one would expect the index to take a rather long time for dropping as much as $s - s_0$. To avoid this risk, the trader would not want to choose a put with strike K such that the level s_0 defining the trade is above the curve $s_0^{max}(\mu_0)$ in Figure 3 for a range of ‘larger’ values of μ_0 .

The alternative to this unsuitable choice of K would be to choose a put with strike K such that the level s_0 stays below the curve $s_0^{max}(\mu_0)$ in Figure 3 for all ‘larger’ values of μ_0 . This alternative refers to the remaining case-(iv)-strategy, and we return to Figure 5 to discuss this strategy in more detail.

Recall that $\mu_0 = 30\%$ and $s_0 = 15000$ in our example. According to the function $b_*(s_0) - s_0$ shown in Figure 4, the gap between s_0 and b_* in our example is about 30 points of the Dow Jones index. Clearly, if $s - s_0$ is of the order of 50 points and the value s of the Dow Jones is of the order of 15000 points, then a drop from s to s_0 would not have any effect on the volatility of the index. So, at least in our example, to be in agreement with the model’s assumptions, the value s should be well above b_* (i.e. $b_* \ll s$).

Figure 5 can now be used to illustrate the two different strategies depending on how much the present value s is above b_* . When assuming $b_* < s < b_0$, the trader would sell/exercise immediately, while, when assuming $b_0 < s$, the trader waits for the index to reach b_0 and would then sell/exercise. By the same reason given in Remark 3.2, the trader would not want to wait for the index to reach b_0 if μ_0 is as large as 30%. Therefore, a further but final constraint on where the present value s should be located is $b_* \ll s < b_0$. Note that $b_0 - b_*$ is of the order of 800 points in our example which is on the right scale for taking into account a possible leverage effect if the index drops from s satisfying $b_* \ll s < b_0$ to a level s_0 below $b_* = 15030$. For given s and s_0 , the relation $b_*(\mu_0, s_0) < s < b_0(\mu_0)$ wanted for all ‘larger’ values of μ_0 can be achieved by choosing an appropriate strike level K .

3.1 Choosing the Strike Level

The following steps present a summary of the previous discussion on how to choose the strike level of the put depending on both the stopping levels b_0, b_1, b_* given by Theorem 1.6 and the level s_0^{max} introduced in the paragraph preceding Figure 3 above. After this summary we briefly describe how the strategies given in Theorem 1.6 could be used for trading.

Step 1: Fix a discount rate α and choose an index with present value s . Find σ_0 by comparing implied volatilities calculated from a range of traded options on the index. Decide about the size of $s - s_0$ the index is expected to drop in the near future. Based on analysing historical data or otherwise,

decide about the size of the ‘excited’ volatility σ_1 . Analysing historical data or otherwise, find the average time span of an ‘excited’ volatility regime after a drop of size $s - s_0$ of the index of your choice, that is, find $1/\lambda$. Set $\mu_1 = 0$.

Step 2: For different values of K calculate: $b_1; \tilde{\mu}_0$ such that $b_1 = b_0(\tilde{\mu}_0)$; $s_0^{max}(\tilde{\mu}_0 + \rho_0)$ for sufficiently large ρ_0 ; $b_0(\tilde{\mu}_0 + \rho_0)$; $b_*(\tilde{\mu}_0 + \rho_0, s_0)$. For the right tuning of ρ_0 compare with both Figure 3 where $\tilde{\mu}_0$ and $\tilde{\mu}_0 + \rho_0$ were 13.7% and 30%, respectively, and the comments in Remark 3.1(iii) about the magnitude of 13.7%.

Step 3: Finally choose a put with strike level K such that $b_1 < s_0 < s_0^{max}(\tilde{\mu}_0 + \rho_0)$ and $b_*(\tilde{\mu}_0 + \rho_0, s_0) < s < b_0(\tilde{\mu}_0 + \rho_0)$. We think that, for trading, the present value s of the index and the drop-to-level s_0 would be placed best if the size of $b_*(\tilde{\mu}_0 + \rho_0, s_0) - s_0$ is on a smaller scale than $s - s_0$ as in our example above. Note that this would also entail $b_*(\tilde{\mu}_0 + \rho_0, s_0) \ll s$.

When trading a put of the above choice using the strategies given by Theorem 1.6, assuming that the value of μ_0 is sufficiently negative to be conform with a drop of size $s - s_0$ in the near future, the trader would initially follow the case (iii)(b) strategy.

First, if the level s_0 is reached within the expected time frame, the trader would continue following the case (iii)(b) strategy to the end. In practical terms, the exponential waiting time should be realised by waiting a multiple of the average waiting time $1/\lambda$ where the choice of the multiple is up to the trader.

Second, if the level s_0 is not reached within the expected time frame, the trader would have gained enough new market data to update the value of μ_0 . Based on statistical testing or otherwise, they should decide whether the updated value of μ_0 is below $\tilde{\mu}_0$ or above $\tilde{\mu}_0 + \rho_0$.

If the decision is for the updated μ_0 to be below $\tilde{\mu}_0$, the trader could continue following the case (iii)(b) strategy (updating μ_0 again if necessary), but they should also consider to finish the trade as soon as selling/exercising would not result in any losses.

If the decision is for the updated μ_0 to be above $\tilde{\mu}_0 + \rho_0$, the trader should change to the case (iv) strategy but with respect to the most recent value of the underlying asset. If this new present value s is above $b_*(\tilde{\mu}_0 + \rho_0, s_0)$, they should sell/exercise immediately. However, if it is in the range of s_0 to $b_*(\tilde{\mu}_0 + \rho_0, s_0)$, the trader could continue following the primary case (iii)(b) strategy, unless the level $b_*(\tilde{\mu}_0 + \rho_0, s_0)$ is reached before the drop-to-level s_0 when they should sell/exercise immediately.

Appendix

We verify that the explicit expression given for $V(\cdot, 1, 1)$ in Theorem 1.6(ii) is indeed the value function. Our method of verification is going to be different to the standard method mentioned in Remark 1.7(iii).

First, we introduce the formal differential operator

$$L_1 f(s) = \mu_1 s f'(s) + \frac{1}{2} \sigma_1^2 s^2 f''(s) + \lambda ((K - s)^+ - f(s))$$

and remark that, by standard arguments (see Section 5.2.1 in [10] for example), the function $V(\cdot, 1, 1)$ is continuous and satisfies the variational inequality

$$\min((\alpha I - L_1)V(\cdot, 1, 1), V(\cdot, 1, 1) - (K - \cdot)^+) = 0 \quad \text{on} \quad (0, \infty)$$

in viscosity sense.

Second, given on a family of probability spaces $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}_s, s > 0)$, we consider the Feller process $(\tilde{S}_t, t \geq 0)$ whose generator is the closure of

$$\tilde{L}_1 f = \mu_1 s f' + \frac{1}{2} \sigma_1^2 s^2 f'', \quad f \in C_b^2((0, \infty)),$$

and define the value function

$$\tilde{V}_1(s) = \sup_{\tilde{\tau} \geq 0} \tilde{\mathbf{E}}_s [e^{-\tilde{\beta}\tilde{\tau}} (K - \tilde{S}_{\tilde{\tau}})^+ + \lambda \int_0^{\tilde{\tau}} e^{-\tilde{\beta}u} (K - \tilde{S}_u)^+ du]$$

where $\tilde{\beta} = \alpha + \lambda$ and the supremum is taken over all stopping times with respect to $(\tilde{S}_t, t \geq 0)$.

Third, by [10, Thm.5.2.1], the value function \tilde{V}_1 is the unique viscosity solution to the variational inequality

$$\min \left((\tilde{\beta}I - \tilde{L}_1)f - \lambda(K - \cdot)^+, f - (K - \cdot)^+ \right) = 0 \quad \text{on} \quad (0, \infty)$$

satisfying a linear growth condition. Therefore, since

$$\alpha I - L_1 = \tilde{\beta}I - \tilde{L}_1 - \lambda(K - \cdot)^+$$

and since $V(\cdot, 1, 1)$ is bounded, it follows from the above first step that $V(\cdot, 1, 1) = \tilde{V}_1$.

Fourth, by [10, Lemma 5.2.2] (uniform ellipticity on compact subsets of $(0, \infty)$ is sufficient in our case), the value function \tilde{V}_1 is C^2 inside the continuation region, and, by [10, Prop.5.2.1], it is C^1 on the boundary. As a consequence, the value function $V(\cdot, 1, 1) = \tilde{V}_1$ is C^1 on the whole domain $(0, \infty)$ and C^2 inside of $\{s > 0 : V(\cdot, 1, 1) > (K - s)^+\}$.

If one can now prove that the set $\{s > 0 : V(\cdot, 1, 1) > (K - s)^+\}$ must have the form (b_1, ∞) for some optimal stopping level $b_1 \in (0, K)$, then $V(\cdot, 1, 1)$ would satisfy both the boundary and pasting conditions (2.2) and, in classical sense, the equations above (2.2) on page 8. But the latter must have the solution given by (2.3) leading to the explicit expression given for $V(\cdot, 1, 1)$ in Theorem 1.6(ii).

Remark A.1. The optimal stopping level b_1 must satisfy (2.4) because $V(\cdot, 1, 1)$ satisfies (2.2). However, finding b_1 by solving (2.4) requires showing uniqueness of solutions to a non-linear equation. This uniqueness problem was neither addressed in [3] nor in [4]. For completeness, we are going to show uniqueness of solutions to (2.4) in Lemma A.3 after the next lemma. However, Example A.4 at the end of the Appendix shows that this uniqueness is not an intrinsic property of equations like (2.4) even if the equations were derived from an optimal stopping problem with convex gain function.

On the whole, finishing our method of verification, we only need to prove the following lemma.

Lemma A.2. There exists $b_1 \in (0, K)$ such that

$$\{s > 0 : V(\cdot, 1, 1) > (K - s)^+\} = (b_1, \infty).$$

Proof. The proof can be divided into the two cases $\mu_1 \leq \alpha$ and $\mu_1 > \alpha$.

For $\mu_1 \leq \alpha$, the process $(e^{-\alpha t} S_t, t \geq 0)$ is a $\mathbf{P}_{s,1,1}$ -supermartingale and the conclusion of the lemma follows by copying the proof of Proposition 1 in [4].

For $\mu_1 > \alpha$, we present a proof which works for $\mu_1 \geq 0$.

First, since our gain function is $(K - \cdot)^+$, one has for all μ_1 that $[K, \infty) \times \{1\} \times \{1\}$ is in the continuation region. As a consequence,

$$b_1 \stackrel{\text{def}}{=} \sup\{s > 0 : V(s, 1, 1) = K - s\} < K$$

because the stopping region is closed when both value function and gain function are continuous.

We now want to prove by contradiction that $b_1 > 0$ and that $V(s, 1, 1) = K - s$ for all $s \in (0, b_1]$ finishing the proof of the lemma.

Recall the regularity properties of $V(\cdot, 1, 1)$ discussed in the first part of the Appendix. So, if $b_1 = 0$, that is, if the supremum is taken over the empty set, then

$$V(s, 1, 1) = c_1 s^{\beta^+} + c_2 s^{\beta^-} + h(s) \quad \text{for } s \in (0, K)$$

by the same arguments which led to (2.3) in Section 2.1. Since $V(\cdot, 1, 1)$ is bounded, the coefficient c_2 would have to be zero, and hence, using the explicit choices for h given in Remark 1.7(ii),

$$\lim_{s \downarrow 0} V(s, 1, 1) = \frac{\lambda K}{\alpha + \lambda} < K,$$

which would contradict $V(s, 1, 1) \geq (K - s)^+$ for all $s > 0$.

Next, assume $b_1 > 0$ and $V(s, 1, 1) > K - s$ for some $s \in (0, b_1]$.

Then there would exist a component (u_1, u_2) of the continuation region, where⁴ $0 < u_1 < u_2 < b_1$, such that

$$\mu_1 s \partial_1 V(s, 1, 1) + \frac{1}{2} \sigma_1^2 s^2 \partial_{11} V(s, 1, 1) + \lambda(K - s) - (\alpha + \lambda)V(s, 1, 1) = 0 \quad (\text{A.1})$$

for all $s \in (u_1, u_2)$ in classical sense and

$$V(u_1, 1, 1) = K - u_1, \quad \partial_1 V(u_1, 1, 1) = -1, \quad V(u_2, 1, 1) = K - u_2, \quad \partial_1 V(u_2, 1, 1) = -1. \quad (\text{A.2})$$

Note that, since $V(\cdot, 1, 1)$ dominates $(K - \cdot)^+$, (A.1) implies

$$\partial_{11} V(s, 1, 1) \geq -\frac{2\mu_1}{s\sigma_1^2} \partial_1 V(s, 1, 1) + \frac{2\alpha}{s^2\sigma_1^2} V(s, 1, 1) \quad \text{for } s \in (u_1, u_2)$$

leading to

$$\partial_{11} V(s, 1, 1) \geq \frac{2\alpha}{u_2^2 \sigma_1^2} V(s, 1, 1) \geq \frac{2\alpha}{u_2^2 \sigma_1^2} (K - u_2) > 0 \quad \text{for } s \in U_2 \cap (u_1, u_2), \quad (\text{A.3})$$

because $\mu_1 \geq 0$ and, by continuity, $\partial_1 V(\cdot, 1, 1)$ is negative in a neighbourhood U_2 of u_2 .

Thus, $\partial_1 V(\cdot, 1, 1)$ is strictly increasing in a left neighbourhood of u_2 so that, for (A.2) to be true, there must exist a largest inflection point $u_0 \in (u_1, u_2)$ given by

$$u_0 = \sup\{s \in (u_1, u_2) : \partial_{11} V(s, 1, 1) = 0\}.$$

Of course, by continuity of $\partial_{11} V(\cdot, 1, 1)$ in the continuity region, $\partial_{11} V(u_0, 1, 1) = 0$, and, since $\partial_1 V(\cdot, 1, 1)$ is strictly increasing on (u_0, u_1) , $\partial_1 V(u_0, 1, 1) < -1$. Hence, by the same arguments leading to (A.3),

$$\partial_{11} V(u_0, 1, 1) \geq \frac{2\alpha}{u_2^2 \sigma_1^2} (K - u_2) > 0$$

which contradicts $\partial_{11} V(u_0, 1, 1) = 0$. □

Lemma A.3. There is exactly one solution $(c_1, c_2, d_2, b_1) \in \mathbb{R}^3 \times (0, K)$ to the system (2.4).

⁴One can rule out $u_1 = 0$ the same way $b_1 = 0$ was ruled out.

Proof. We only show the lemma in the case of $\alpha + \lambda \neq \mu_1$ using the corresponding function h given in Remark 1.7(ii) because nothing fundamental changes when doing the calculation in the remaining single case of $\alpha + \lambda = \mu_1$ with another function h .

First, we ignore the last equation of the system (2.4) and replace b_1 by an arbitrary $b > 0$. The resulting system of equations reads

$$\begin{aligned} c_1 K^{\beta^+} + c_2 K^{\beta^-} - \frac{\lambda K}{\alpha + \lambda - \mu_1} + \frac{\lambda K}{\alpha + \lambda} &= d_2 K^{\beta^-}, \\ c_1 \beta^+ K^{\beta^+} + c_2 \beta^- K^{\beta^-} - \frac{\lambda K}{\alpha + \lambda - \mu_1} &= d_2 \beta^- K^{\beta^-}, \\ K - b &= c_1 b^{\beta^+} + c_2 b^{\beta^-} - \frac{\lambda b}{\alpha + \lambda - \mu_1} + \frac{\lambda K}{\alpha + \lambda}, \end{aligned}$$

and, for each $b > 0$, this system admits a unique solution c_1, c_2, d_2 .

To analyse the non-linear last equation of the system (2.4), we only need to know c_1 and c_2 which are explicitly given by

$$\begin{aligned} c_1 &= \frac{\lambda K^{1-\beta^+}}{(\beta^- - \beta^+)} \left[\frac{\beta^-}{\alpha + \lambda - \mu_1} - \frac{\beta^-}{\alpha + \lambda} - \frac{1}{\alpha + \lambda - \mu_1} \right], \\ c_2(b) &= \left[K - b + \frac{\lambda b}{\alpha + \lambda - \mu_1} - \frac{\lambda K}{\alpha + \lambda} - c_1 b^{\beta^+} \right] \frac{1}{b^{\beta^-}}. \end{aligned}$$

Second, for each $b > 0$, we introduce the function

$$V_b(s) = c_1 s^{\beta^+} + c_2(b) s^{\beta^-} - \frac{\lambda s}{\alpha + \lambda - \mu_1} + \frac{\lambda K}{\alpha + \lambda}, \quad s > 0,$$

and remark that $b > 0$ satisfies

$$-b = c_1 \beta^+ b^{\beta^+} + c_2(b) \beta^- b^{\beta^-} - \frac{\lambda b}{\alpha + \lambda - \mu_1}$$

if and only if

$$\frac{d}{ds} V_b(s)|_{s=b} = V'_b(b) = -1.$$

Therefore, when setting $\Gamma(b) = V'_b(b)$ for $b > 0$, the proof of the lemma reduces to showing that the equation $\Gamma(b) = -1$ has exactly one root between zero and K ; and this will be shown next.

By straight forward calculation, we have that

$$\Gamma(b) = c_1 b^{\beta^+-1} (\beta^+ - \beta^-) + \frac{\alpha K \beta^-}{(\alpha + \lambda) b} - \frac{(\alpha - \mu_1) \beta^- + \lambda}{\alpha + \lambda - \mu_1} \quad \text{for } b > 0$$

so that

$$\lim_{b \rightarrow 0} \Gamma(b) = -\infty \quad (\text{since } \beta^- < 0) \quad \text{and} \quad \Gamma(K) = 0.$$

Thus, by Intermediate Value Theorem, there exists $b_1 \in (0, K)$ such that $\Gamma(b_1) = -1$.

For uniqueness, one only has to show that $\Gamma(\cdot)$ is increasing on $(0, K)$, that is, $\Gamma'(b) \neq 0$ for all $b \in (0, K)$.

But,

$$\Gamma'(b) = c_1(\beta^+ - 1)(\beta^+ - \beta^-)b^{\beta^+ - 2} - \frac{\alpha K \beta^-}{\alpha + \lambda} b^{-2},$$

and hence, for $b \in (0, K)$, the equality $\Gamma'(b) = 0$ is equivalent to

$$\left(\frac{b}{K}\right)^{\beta^+} = \frac{\alpha(\alpha + \lambda - \mu_1)}{\frac{1}{2}\sigma_1^2\lambda(\beta^+ - 1)(\beta^- - 1)},$$

where we have used that β^- is a root of the equation (1.6). As the above right-hand side is always negative because $\beta^+ > 1$ if $\alpha + \lambda > \mu_1$ and $\beta^+ < 1$ if $\alpha + \lambda < \mu_1$, there is no $b > 0$ such that $\Gamma'(b) = 0$. \square

Example A.4. Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a bounded smooth convex function satisfying

$$g(x) = \begin{cases} 4 & : x = 1/2 \\ 1 & : x = 1 \end{cases}, \quad g'(x) = \begin{cases} -8 & : x = 1/2 \\ -1 & : x = 1 \end{cases}, \quad g(x) = 0, x \geq 3,$$

and consider the value function $V(x) = \sup_{\tau \geq 0} \mathbf{E}[e^{-\tau} g(X_\tau^x)]$ where $X_t^x = xe^{\sqrt{2}B_t}$, $t \geq 0$, is a geometric Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Following the method used in [3, 4], the free-boundary value problem associated with this optimal stopping problem is

$$0 = xV'(x) + x^2V''(x) - V(x) \quad \text{for } x > x_0$$

subject to

$$V(x_0) = g(x_0), \quad V'(x_0) = g'(x_0), \quad \lim_{x \rightarrow \infty} V(x) = 0.$$

As any solution to this problem must have the form $c_1x + c_2x^{-1}$, the above boundary and pasting conditions result in $c_1 = 0$ and two equations

$$g(x_0) = c_2x_0^{-1}, \quad g'(x_0) = -c_2x_0^{-2}$$

for the pair of unknowns (c_2, x_0) .

There are at least two solutions to these equations, $(c_2, x_0) = (1, 1)$ and $(c_2, x_0) = (2, 1/2)$, but there might be even more. Note that the value function is unique and can only be identical to one of the candidate value functions build from these solutions (c_2, x_0) . Hence this example indeed justifies Remark A.1 on page 24.

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